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FAST TRACK COMMUNICATION

A classical bound on quantum entropy

Cosmas K Zachos

High Energy Physics Division, Argonne National Laboratory, Argonne, IL 60439-4815, USA

E-mail: zachos@hep.anl.gov

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Abstract
A classical upper bound for quantum entropy is identified and illustrated, $0 \leqslant S_q \leqslant \ln(e\sigma^2/2\hbar)$, involving the variance σ^2 in phase space of the classical limit distribution of a given system. *A fortiori*, this further bounds the corresponding information-theoretical generalizations of the quantum entropy proposed by Rényi.

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1. Introduction

Recurrent problems in four-dimensional BPS black holes focus on the entropic behaviour of the respective complex structure moduli spaces, and, perhaps independently, on the corresponding holographic entanglement information lost in decoherence, and associated Hawking radiation paradoxes [1]. They all rely on the fundamental and dependable statistical concept of entropy, which accounts collectively for the flow of information in these systems, and for which robust estimates are needed, in lieu of detailed accounts of quantum states. Ideally, such estimates would only require gross geometrical and semiclassical features of the system involved and ignore quantum mechanical interference subtleties.

Classical continuous distributions have been studied in probability and information theory for quite a long time, and Shannon [2] has derived handy upper bounds for their entropy, and thus crude least information estimates, in the 1940s. Approximate counting of quantum microstates, however, is normally toilsome and can be approximated heuristically by semiclassical proposals [3], which, ultimately, should devolve to a bona fide classical limit, despite occasional ambiguities and complications along the way [4]. However, a more systematic approach was initiated by Braunss [5], who appreciated the underlying simplicity of phase space in taking a classical limit of intricate quantum systems. He thus tracked the information loss involved in smearing away quantum effects, to argue that the entropy of a quantum system is majorized by that of its classical limit, as $\hbar$-information of the former is forfeited in the latter, an intuitively plausible relation.

The purpose of the present communication is to simply combine the two inequalities into a general upper bound of the quantum entropy of a system provided essentially by just the logarithm of the variance in phase space of the classical limit distribution of that system. The resulting inequality, equation (9), is illustrated simply by the elementary physics paradigm of a thermal bath of oscillator excitations of one degree of freedom, whose phase-space representation is an obvious maximal entropy Gaussian.

Note that there is no specific assumption of a particular spectral behaviour—or even of the existence of a Hamiltonian—for the systems covered by the inequality. Extension to arbitrary degrees of freedom and tighter bounds contingent on the circumstances of detailed physical applications are conceptually straightforward, even though specific application to the moduli phase spaces or holographic entanglement of black holes is reserved for a future, less general, report.

In passing, and because it fits naturally with the computational technique involved, the corresponding quantum Rényi entropies [6] are also evaluated explicitly here for the same prototype system, to illustrate the broad fact that these entropies are majorized by the Gibbs–Boltzmann entropy, and thus also by the bound discussed here. Rényi-generalized entropies were originally introduced as a measure of complexity in optimal coding theory [6], and have been applied to turbulence, chaos and fractal systems, as well as semi-inclusive multiparticle production [7, 8]; however, apparently, they have not attained significance in black hole physics yet, nor in current noncommutative geometry efforts.

2. Shannon and Boltzmann–Gibbs entropy in phase space

For a continuous distribution function $f(x, p)$ in phase space, the classical (Shannon information) entropy is

$$S_{\mathrm{cl}} = -\int \mathrm{d}x\,\mathrm{d}p\, f \ln(f). \tag{1}$$

For a given distribution function $f(x, p)$, without loss of generality centred at the origin, normalized, $\int \mathrm{d}x\,\mathrm{d}p\, f = 1$, and with a given variance, $\sigma^2 = \langle x^2 + p^2\rangle = \int \mathrm{d}x\,\mathrm{d}p(x^2 + p^2) f$, it is evident from elementary constrained variation of this $S_{\mathrm{cl}}[f]$ w.r.t. f [2] (also see [9]), that it is maximized by the Gaussian, $f_g = \exp(-(x^2 + p^2)/\sigma^2)/\sigma^2\pi$, to $S_{\mathrm{cl}} = 1 + \ln(\pi\sigma^2)$.

That is, a Gaussian represents maximal disorder and minimal information—in thermodynamics, least dispersal energy would be available.

Thus, it leads to a standard result in information theory [2], Shannon's inequality,

$$S_{\mathrm{cl}} \leqslant \ln(\pi e \sigma^2), \tag{2}$$

which provides an upper bound on the lack of information in such distributions.

Note that, in general, S_{cl} is unbounded above, as it diverges for delocalized distributions, $\sigma \to \infty$, containing no information. In contrast to the Boltzmann–Gibbs entropy, it is also unbounded below, given ultralocalized peaked distributions ($\sigma \to 0$), which reflect complete order and information.

In quantum mechanics, the sum over all states is given by the standard von Neumann entropy [10] for a density matrix ρ,

$$0 \leqslant S_q = -\mathrm{Tr}\,\rho \ln \rho = -\langle \ln \rho\rangle. \tag{3}$$

This transcribes in phase space [5, 11] through the Wigner transition map [12] to

$$0 \leqslant S_q = -\int \mathrm{d}x\,\mathrm{d}p\, f \ln_\star(hf), \tag{4}$$

where the $\star$-product [11]

$$\star \equiv \exp\left(\frac{\mathrm{i}\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)\right) \tag{5}$$

serves to define $\star$-functions, such as the $\star$-logarithm, above, e.g. through $\star$-power expansions,

$$\ln_\star(hf) \equiv -\sum_{n=1}^{\infty} \frac{(1-hf)^n_\star}{n}. \tag{6}$$

Braunss [5] has argued that, for S_{cl} defined by $S_q + \ln h$ in the limit that the Planck constant $\hbar \to 0$,

$$0 \leqslant S_q \leqslant S_{\mathrm{cl}} - \ln h. \tag{7}$$

The logarithmic offset term relying on the Planck constant h accounts for the scale [3] of the phase-space area element $\mathrm{d}x\,\mathrm{d}p$ in (4). This scale, h, should divide $\mathrm{d}x\,\mathrm{d}p$ to yield a dimensionless phase-space cell. Correspondingly, it should then multiply f, to preserve 'probability', $\int \mathrm{d}x\,\mathrm{d}p\, f = 1$, in the Wigner transition map from the density matrix ρ to the Wigner function f. For example, for a pure state [12],

$$f(x,p) = \frac{1}{h}\int \mathrm{d}y\,\psi^*\left(x - \frac{1}{2}y\right)\mathrm{e}^{-\mathrm{i}yp/\hbar}\psi\left(x+\frac{1}{2}y\right). \tag{8}$$

The classical limit normally entails variations of phase-space variables on scales much larger than $\hbar$. Therefore, these variables are normally scaled down to scales matched to such activity. As illustrated explicitly in the next section, comparing quantum and classical entropies relies on the above offset. The upper bound in this Braunss inequality reflects the loss of quantum information involved in the smearing implicit in the classical limit[1], effectively regarded as an extreme limit of subadditivity [3].

Combined with Shannon's bound, this now amounts to

$$\boxed{0 \leqslant S_q \leqslant \ln\left(\frac{e\sigma^2}{2\hbar}\right)}, \tag{9}$$

i.e., the entropy is bounded above by an expression involving the variance of the corresponding classical limit distribution function. It readily generalizes to multidimensional phase space (R^{2N}, in which case the logarithm is evidently multiplied by N, in evocation of Bekenstein's bound), and contexts where more information (e.g., on asymmetric variances) happens to be available, or refinement desired.

By virtue of (6), the quantum entropy is recognized as an expansion

$$S_q = \sum_{n=1}^{\infty}\frac{\langle(1-\rho)^n\rangle}{n} = \sum_{n=1}^{\infty}\frac{\langle(1-hf)^n_\star\rangle}{n}. \tag{10}$$

The leading term, $n = 1$, $1 - \mathrm{Tr}\,\rho^2 = \langle 1 - hf\rangle$, is the *impurity* [10–12], often referred to as linear entropy. Like the entropy itself, it vanishes for a pure state [10–12], for which $\rho^2 = \rho$,

[1] Readers unfamiliar with the classical limit might find loss of the quantum uncertainty of the theory counterintuitive and discordant with the loss of information involved. Actually, *the resolution to access the uncertainty* is sacrificed in this limit. A standard consequence of the Cauchy–Schwarz inequality for Wigner functions is $|f| \leqslant 2/h$, [12], reflecting the uncertainty principle: the impossibility of localizing f in phase space, through a delta function. The best one can do is to take a pillbox cylinder of base $h/2$ and height $2/h$, properly normalized to $1 = \int \mathrm{d}x\,\mathrm{d}p\, f$. Now, scaling the phase-space variables down and f up (to preserve this normalization—the volume of the pillbox, as in the above discussion of the offset) ultimately collapses the base of the pillbox to a mere point in phase space and leads to a divergent height for f, a delta function, characteristic of a perfectly localized classical particle. However, several different quantum configurations will reduce to this same limit: it is this extra quantum information on h-dependent features, e.g. interference, that is obliterated in the limit.

or, equivalently, $f \star f = f/h$. Each term in the above expansion then projects out ρ, or $\star hf$, respectively: *pure states saturate the lower bound on* S_q.

A likewise additive (extensive) generalization of the quantum entropy is the Rényi entropy [6],

$$R_\alpha = \frac{1}{1-\alpha}\ln\langle\rho^{\alpha-1}\rangle = \frac{1}{1-\alpha}\ln\int\frac{\mathrm{d}x\,\mathrm{d}p}{h}(hf)^\alpha_\star, \tag{11}$$

where the limit $\alpha \to 1$ yields $R_1 = S_q$, and the above-mentioned impurity is $1-\exp(-R_2)$. For continuous distributions (infinity of components) discussed here, R_0 is divergent.

For $\alpha \geqslant 1$, $R_\alpha \geqslant R_{\alpha+1}$, so $S_q \geqslant R_\alpha$, and it is also bounded below by 0 [6], i.e.,

$$S_q \geqslant R_\alpha \geqslant R_{\alpha+1} \geqslant 0, \tag{12}$$

so that, *a fortiori*, the Rényi entropy is also bounded by (9).

3. Gaussian illustration

To illustrate the above inequalities, consider the Gaussian Wigner function of *arbitrary* half-variance E,

$$f(x,p,E) = \frac{\exp\left(-\frac{x^2+p^2}{2E}\right)}{2\pi E} = \exp\left(-\frac{x^2+p^2}{2E} - \ln(2\pi E)\right). \tag{13}$$

This happens to be the phase-space Wigner transform of a Maxwell–Boltzmann thermal distribution for a harmonic oscillator [13], in suitably rescaled units, normalized properly to unity, and with mean energy $E = \langle(x^2+p^2)/2\rangle$.

Calculation of the entropy of this distribution, is, of course, a freshman physics problem, but its independent phase-space derivation [14] (also see [15]), is reviewed here, i.e., evaluation of (4) directly.

For $E = \hbar/2$, the distribution reduces to just f_0, the Wigner function for a pure state (the ground state of the harmonic oscillator). Hence [11, 12],

$$f_0 \star f_0 = \frac{f_0}{h}, \tag{14}$$

so that f_0 is $\star$-orthogonal to each of the terms in the sum (6), and hence $S_q = 0$, indicating saturation of the maximum possible information content.

For generic width E, the Wigner function f is not that of a pure state, but it still happens to always amount to a $\star$-exponential [16] ($\mathrm{e}^a_\star \equiv 1 + a + a\star a/2! + a\star a\star a/3! + \cdots$) as well,

$$hf = \mathrm{e}^{-\frac{x^2+p^2}{2E}+\ln(\hbar/E)} = \mathrm{e}_\star^{-\frac{\beta}{2\hbar}(x^2+p^2)+\ln\left(\frac{\hbar}{E}\cosh(\beta/\hbar)\right)}, \tag{15}$$

where an 'inverse temperature' variable $\beta(E,\hbar)$ is useful to define

$$\tanh(\beta/2) \equiv \frac{\hbar}{2E} \leqslant 1 \quad\Longrightarrow\quad \beta = \ln\frac{E+\hbar/2}{E-\hbar/2}. \tag{16}$$

(Thus the above pure state f_0 corresponds to zero temperature, $\beta = \infty$.)

Since $\star$-functions, by virtue of their $\star$-expansions, obey the same functional relations as their non-$\star$ analogues, inverting the $\star$-exponential through the $\star$-logarithm and integrating (4) yields directly the standard thermal physics result,

$$S_q(E,\hbar) = \frac{E}{\hbar}\ln\left(\frac{2E+\hbar}{2E-\hbar}\right) + \frac{1}{2}\ln\left(\left(\frac{E}{\hbar}\right)^2 - \frac{1}{4}\right) = \frac{\beta}{2}\coth(\beta/2) - \ln(2\sinh(\beta/2)). \tag{17}$$

Indeed, this can be seen to be a monotonically nondecreasing function of E, attaining the lower bound 0 for the pure state $E \to \hbar/2$ ($\beta \to \infty$, zero temperature).

The classical limit, $\hbar \to 0$ ($\beta \to 0$, infinite temperature) thus follows,

$$S_q \to 1 + \ln(E/\hbar) = \ln(\pi e 2E) - \ln h = S_{\rm cl}(E) - \ln h, \tag{18}$$

and is explicitly seen to bound expression (17) for all E, saturating it for large $E \gg \hbar$, in accordance with Braunss' bound. That is, the upper bound (9) is saturated for Gaussian quantum Wigner functions with $\sigma^2 \gg \hbar$.

Note the region $E < \hbar/2$, corresponding to ultralocalized spikes excluded by the uncertainty principle, was not allowed by the above derivation method, since, in this region, no $\star$-Gaussian can be found to represent the Gaussian. (It would amount to complex β and S_q, linked to thermal expectations of the oscillator parity operator.)

Note. An alternate heuristic proposal of [3] for the classical limit of the entropy effectively starts from the Husimi phase-space representation [12]; it first effectively drops all $\star_H$s in (4) and easily evaluates (1) instead (which is well defined because $f_H \geqslant 0$ automatically), *before* completing the transition to the classical limit $\hbar \to 0$. It also, ultimately, yields the same answer (18), since the Husimi representation of the Gaussian Wigner function (13),

$$f_H \equiv \int {\rm d}x'\,{\rm d}p' \frac{\exp(-((x'-x)^2 + (p'-p)^2)/\hbar)}{\pi\hbar} f(x', p') = \frac{\exp\left(-\frac{x^2+p^2}{2E+\hbar}\right)}{\pi(2E+\hbar)}, \tag{19}$$

is also a Gaussian. Utilized to evaluate (1), it yields $\ln(\pi e(2E+\hbar))$, which has the more direct expression $S_{\rm cl}$ of (18) as its classical limit. (For the ground state, $E = \hbar/2$, which is a coherent state, this semiclassical entropy reduces to a characteristic minimal value, $1 + \ln h$.)

By virtue of (15), $\star$-powers of the Gaussian are also straightforward to take, and thus the Rényi entropies can be readily computed:

$$R_\alpha = \frac{1}{1-\alpha} \ln\left(\frac{(2\sinh(\beta/2))^\alpha}{2\sinh(\alpha\beta/2))}\right) = \frac{1}{\alpha-1} \ln\left(\left(\frac{E}{\hbar} + \frac{1}{2}\right)^\alpha - \left(\frac{E}{\hbar} - \frac{1}{2}\right)^\alpha\right). \tag{20}$$

Note $\alpha \to 1$ checks with the above (17), $R_1 \to S_q$. Also, in the pure state limit, $E = \hbar/2$, it is evident that $R_\alpha = 0$ checks for all $\alpha \geqslant 1$. (For $\alpha > 1$ and the small disallowed values $E < \hbar/2$, $R_\alpha < 0$.)

R_α is also a nondecreasing function of E, and, in comportance with (12), a nonincreasing function of α. Up to an additive, α-dependent constant, the classical limit is identical to that for the entropy itself,

$$R_\alpha \to \frac{\ln\alpha}{\alpha - 1} + \ln(E/\hbar), \tag{21}$$

in agreement with the classical result of [8]. It may well be that, as in the contexts touched upon in the introduction, specific αs may well provide more detailed or practical measures of complexity in Hawking radiation with sparse information available.

If a specific quantum Hamiltonian were actually available for the system in question (a rare occurrence), then the classical limit of the entropy of the system would be straightforward—and thus the inequality discussed here would not be that powerful, since the classical entropy itself would be at hand, in general lower than the Shannon bound.

For such a simple system, the upper-bounding classical entropy would result out of the phase-space partition function specified by the corresponding classical Hamiltonian (the Weyl symbol of the quantum Hamiltonian). This is easily illustrated explicitly by Hamiltonians which are positive Nth powers of the oscillator Hamiltonian, so that, simply,

$$f_{\rm cl} \propto \exp(-((x^2 + p^2)/2E)^N). \tag{22}$$

The bounding classical entropy then reduces by standard thermodynamic evaluation to be just (1),

$$S_{\mathrm{cl}} = \frac{1}{N} + \ln\left(2\pi E \Gamma\left(1 + \frac{1}{N}\right)\right), \tag{23}$$

lower than the corresponding Shannon bound,

$$1 + \ln\left(\pi E \frac{\Gamma(1 + 2/N)}{\Gamma(1 + 1/N)}\right). \tag{24}$$

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Note added. A careful reader has identified a technical gap in Braunss' formal proof of his inequality in [5], which is, nevertheless, assumed here.

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